

Uniqueness and nonuniqueness in inverse hyperbolic problems and the black holes phenomenon.

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July 22, 2014

Abstract

This paper consists of two parts. In the first part we describe the recent works on the inverse problems for the wave equation in $(n + 1)$ -dimensional space equipped with pseudo-Riemannian metric with Lorentz signature. We study the conditions of the existence of black (or white) holes for these wave equations. In the second part we prove energy type estimates on a finite time interval in the presence of black or white holes. We use these estimates to prove the nonuniqueness of the inverse problems.

1 Introduction.

A powerful method for solving the inverse hyperbolic problem for equations of the form $\frac{\partial^2 u}{\partial t^2} + Au = 0$ where A is a Laplace-Beltrani operator with time-independent coefficients, was discovered by M.Belishev more then twenty years ago. It is called the Boundary Control (BC) method. It was further developed by M.Belishev, M.Belishev and Y.Kurylev, Y.Kurylev and M.Lassas and others (see [B1], [B2], [KKL] and further references there). An impotant part of the solution of the hyperbolic inverse problem is played by the unique continuation theorem due to D.Tataru [T]. In [E1], [E2] the author proposed a new approach to the inverse hyperbolic problem that includes ideas from the BC-method. This new method allowed to solve some inverse hyperbolic

problems that were not accessible by the BC-method: In [E3] the case of hyperbolic equations with time-dependent coefficients was considered and in [E4] the case of the hyperbolic equation with general pseudo-Riemannian time-independent metric was treated. In the following sections we describe the main results of [E4] and [E5].

An interesting phenomenon discussed in [E5] is the appearance of black holes. These black holes are called artificial black holes (they are also called acoustic black holes, or optical black holes) to distinguish from the black holes in the general relativity. Artificial black holes attracted a great interest of physicists (see [NVV], [V] and additional references there) because the physisists hope to create and study the black hole in the laboratory and expect that this will help in the understanding of the black holes in the universe.

In the last two sections we prove the energy type estimates on a finite time interval in the presence of black or white holes. We use these estimates to prove the nonuniqueness in the inverse problems.

2 The inverse hyperbolic problems.

Let Ω be a bounded domain in \mathbf{R}^n with smooth boundary $\partial\Omega$. Let $\Gamma \subset \partial\Omega$ be an open subset of $\partial\Omega$.

Consider a hyperbolic equation in the cylinder $\Omega \times \mathbf{R}$:

$$(2.1) \quad \sum_{j,k=0}^n \frac{1}{\sqrt{(-1)^n g(x)}} \frac{\partial}{\partial x_j} \left(\sqrt{(-1)^n g(x)} g^{jk}(x) \frac{\partial u(x_0, x)}{\partial x_k} \right) = 0,$$

where $x = (x_1, \dots, x_n) \in \overline{\Omega}$, $x_0 \in \mathbf{R}$ is the time variable, the coefficients in (2.1) are smooth and independent of x_0 , $[g_{jk}(x)]_{j,k=0}^n = ([g^{jk}(x)]_{j,k=0}^n)^{-1}$ is a pseudo-Riemannian metric with the Lorenz signature, i.e. the quadratic form $\sum_{j,k=0}^n g^{jk}(x) \xi_j \xi_k$ has the signature $(1, -1, -1, \dots, -1)$ for all $x \in \overline{\Omega}$. Here $g(x) = (\det[g^{jk}(x)]_{j,k=0}^n)^{-1}$. Note that $(-1)^n g(x) > 0$, $\forall x \in \overline{\Omega}$.

We assume that

$$(2.2) \quad g^{00}(x) > 0, \quad x \in \overline{\Omega},$$

i.e. $(1, 0, \dots, 0)$ is not a characteristic direction, and that

$$(2.3) \quad \sum_{j,k=1}^n g^{jk}(x) \xi_j \xi_k < 0 \quad \text{for } \forall (\xi_1, \dots, \xi_n) \neq (0, \dots, 0), \quad \forall x \in \overline{\Omega},$$

i.e. the quadratic form (2.3) is negative definite. Note that (2.3) equivalent to the condition that

$$(2.4) \quad g_{00}(x) > 0, \quad x \in \overline{\Omega},$$

i.e. that $(1, 0, \dots, 0)$ is a time-like direction.

We consider the initial-boundary value problem for the equation (2.1) in the cylinder $\Omega \times \mathbf{R}$:

$$(2.5) \quad u(x_0, x) = 0 \quad \text{for} \quad x \in \Omega, \quad x_0 < 0,$$

$$(2.6) \quad u(x_0, x)|_{\partial\Omega \times \mathbf{R}} = f(x_0, x'), \quad x' \in \partial\Omega,$$

where $f(x_0, x')$ has a compact support in $\partial\Omega \times \mathbf{R}$.

Let Λf be the Dirichlet-toNeumann (DN) operator, i.e.

$$(2.7) \quad \Lambda f = \sum_{j,k=0}^n g^{jk}(x) \frac{\partial u}{\partial x_j} \nu_k(x) \left(- \sum_{p,r=1}^n g^{pr}(x) \nu_p \nu_r \right)^{-\frac{1}{2}} \Big|_{\partial\Omega \times \mathbf{R}},$$

where $\nu_0 = 0, (\nu_1, \dots, \nu_n)$ is the unit outward normal vector to $\partial\Omega \subset \mathbf{R}^n$, $u(x_0, x)$ is the solution of (2.1), (2.5), (2.6).

Consider a smooth change of variables of the form:

$$(2.8) \quad \begin{aligned} \hat{x}_0 &= x_0 + a(x), \\ \hat{x} &= \varphi(x), \end{aligned}$$

where $\varphi(x)$ is a diffeomorphism of $\overline{\Omega}$ onto some domain $\hat{\Omega}$ such that $\overline{\Gamma} \subset \partial\hat{\Omega}$, $\varphi(x) = x$ on $\overline{\Gamma}$, $a(x) = 0$ on $\overline{\Gamma}$. Note that (2.8) is an identity map on $\overline{\Gamma} \times \mathbf{R}$. Note also that the map (2.8) transforms (2.1) into an equation of the same form in $\hat{\Omega} \times \mathbf{R}$.

The following theorem holds:

Theorem 2.1. (c.f.[E4]): Let L and \hat{L} be two operators of the form (2.1) in $\Omega \times \mathbf{R}$ and $\hat{\Omega} \times \mathbf{R}$ respectively. Consider initial-boundary value problems of the form (2.5), (2.6) for L and \hat{L} . Suppose $\Lambda f = \hat{\Lambda} f$ on $\Gamma \times \mathbf{R}$ for all $f \in C_0^\infty(\Gamma \times \mathbf{R})$ where $\Lambda, \hat{\Lambda}$ are DN operators for L, \hat{L} , respectively. Suppose that conditions (2.2) and (2.3) hold for L and \hat{L} . Then there exists a map of the form (2.8) such that

$$(2.9) \quad [\hat{g}^{jk}(\hat{x})]_{j,k=0}^n = J^T(x) [g^{jk}(x)]_{j,k=0}^n J(x),$$

where $([\hat{g}^{jk}(\hat{x})]_{j,k=0}^n)^{-1}$ is the metric tensor for \hat{L} and $J(x)$ is the Jacobi matrix of (2.8).

Remark 2.1. It is enough to know the DN operator on $\Gamma \times (0, T_0)$ for some $T_0 > 0$ instead of $\Gamma \times \mathbf{R}$. More precisely, let T_+ be the smallest number such that $D_+(\bar{\Gamma} \times \{x_0 = 0\}) \supset \bar{\Omega} \times \{x_0 = T_+\}$ where $D_+(\bar{\Gamma} \times \{x_0 = 0\})$ is the forward domain of influence of $\bar{\Gamma} \times \{x_0 = 0\}$ corresponding to (2.1). Analogously let T_- be the smallest number such that $D_-(\bar{\Gamma} \times \{x_0 = T_-\}) \supset \bar{\Omega} \times \{x_0 = 0\}$ where $D_-(\bar{\Gamma} \times \{x_0 = T_-\})$ is the backward domain of influence of $\bar{\Gamma} \times \{x_0 = T_-\}$. If $T_0 > T_- + T_+$ then $\Lambda = \hat{\Lambda}$ on $\Gamma \times (0, T_0)$ implies (2.9), i.e. the isometry of metrics $[g_{jk}(x)]$ and $[\hat{g}_{jk}(\hat{x})]$.

3 The equation of the propagation of light in the moving dielectric medium.

In this section we apply Theorem 2.1 to the equation of the propagation of light in the moving medium.

It was discovered by Gordon [G] that the equation of the propagation of light in a moving medium is given by the hyperbolic equation of the form (2.1) with the metric tensor

$$(3.1) \quad g^{jk}(x) = \eta^{jk} + (n^2(x) - 1)v^j(x)v^k(x),$$

$0 \leq j, k \leq n$, $n = 3$, where $[\eta_{jk}] = [\eta^{jk}]^{-1}$ is the Lorentz metric tensor: $\eta^{jk} = 0$ when $j \neq k$, $\eta^{00} = 1$, $\eta^{jj} = -1$, $1 \leq j \leq n$, $x_0 = t$ is the time, $n(x) = \sqrt{\varepsilon(x)\mu(x)}$ is the refraction index, $w(x) = (w_1(x), w_2(x), w_3(x))$ is the velocity of flow,

$$v^{(0)} = \left(1 - \frac{|w|^2}{c^2}\right)^{-\frac{1}{2}}, \quad v^{(j)} = \left(1 - \frac{|w|^2}{c^2}\right)^{-\frac{1}{2}} \frac{w_j(x)}{c}, \quad 1 \leq j \leq 3,$$

is the four-velocity field of the flow, c is the speed of light in the vacuum. We shall call the equation (2.1) with metric (3.1) the Gordon equation.

Let Ω be a smooth domain in \mathbf{R}^n of the form $\Omega = \Omega_0 \setminus \bigcup_{j=1}^m \bar{\Omega}_j$ where Ω_0 is simply-connected, Ω_j , $1 \leq j \leq m$, are smooth domains called obstacles, $\bar{\Omega}_j \subset \Omega_0$, $1 \leq j \leq m$, $\bar{\Omega}_j \cap \bar{\Omega}_k = \emptyset$ when $j \neq k$.

We shall consider the following initial-boundary value problem for the Gordon equation:

$$(3.2) \quad \begin{aligned} u(x_0, x) &= 0 \quad \text{for } x_0 \ll 0, \quad x \in \Omega, \\ u(x_0, x)|_{\partial\Omega_j \times \mathbf{R}} &= 0, \quad 1 \leq j \leq m, \\ u(x_0, x)|_{\partial\Omega_0 \times \mathbf{R}} &= f(x_0, x), \end{aligned}$$

i.e. $\partial\Omega_0 = \Gamma$ in the notations of Theorem 2.1.

Note that the condition (2.2) of §2 is always satisfied since

$$g^{00} = 1 + (n^2 - 1)(v^0)^2 > 0.$$

The condition that any direction $(0, \xi_1, \dots, \xi_n)$ is not characteristic (c.f. condition (2.3)) holds when

$$(3.3) \quad |w(x)|^2 < \frac{c^2}{n^2(x)}.$$

We shall impose some restrictions on the flow $w(x)$. Let $x = x(s)$ be a trajectory of the flow, i.e.

$$\frac{dx}{ds} = w(x(s)), \quad 0 \leq s \leq 1,$$

where $w(x(s)) \neq 0$ for $0 \leq s \leq 1$. We assume the following condition:

- (A) The trajectories that start and end on $\partial\Omega_0$,
or are closed curves in Ω , are dense in $\overline{\Omega}$.

Theorem 3.1 ((c.f. [E5]).) *Let $[g_{jk}(x)]_{j,k=0}^n$ and $[\hat{g}_{jk}(y)]_{j,k=0}^n$ be two Gordon metrics in domains Ω and $\hat{\Omega}$, respectively. Consider two initial-boundary value problems of the form (3.2) in $\Omega \times \mathbf{R}$ and $\hat{\Omega} \times \mathbf{R}$, respectively, where $\Omega = \Omega_0 \setminus \cup_{j=1}^m \overline{\Omega}_j$, $\hat{\Omega} = \Omega_0 \setminus \cup_{j=1}^{\hat{m}} \overline{\hat{\Omega}}_j$. Assume that the refraction indexes n and \hat{n} are constant and that the flow $w(x)$ satisfies the condition (A). Assume also that (3.3) holds for both metrics. Then $\Lambda = \hat{\Lambda}$ on $\partial\Omega_0 \times \mathbf{R}$ implies that $\hat{n} = n$, $\hat{\Omega} = \Omega$ and the flows $\hat{w}(x)$ and $w(x)$ are equal.*

4 The propagation of light in the slowly moving medium.

In this case one drops the terms of order $\frac{|w|^2}{c^2}$. Then the metric tensor has the form:

$$(4.1) \quad g^{jk}(x) = \eta^{jk} \quad \text{for } 1 \leq j, k \leq n, \quad n = 3,$$

$$g^{00}(x) = n^2(x), \quad g^{0j}(x) = g^{j0}(x) = (n^2(x) - 1) \frac{w_j(x)}{c}, \quad 1 \leq j \leq n.$$

The wave equation with metric (4.1) describes the propagation of light in a slowly moving medium. We shall see that the inverse problem for such equation exhibits some nonuniqueness.

Denote $v_j(x) = g^{0j} = g^{j0}$. We say that the flow $v = (v_1, \dots, v_n)$ is a gradient flow if $v(x) = \frac{\partial b(x)}{\partial x}$ where $b(x) \in C^\infty(\overline{\Omega})$, $b(x) = 0$ on $\partial\Omega_0$.

Theorem 4.1. (c.f. [E4]) *Consider two initial-boundary value problems in domains $\Omega \times \mathbf{R}$ and $\hat{\Omega} \times \mathbf{R}$ for operators of the form (2.1) with metrics $[g^{jk}(x)]$, $[\hat{g}^{jk}(\hat{x})]$ of the form (4.1). Assume that the DN operators Λ and $\hat{\Lambda}$ are equal on $\partial\Omega_0 \times \mathbf{R}$. Assume that there exists an open connected and dense $O \subset \Omega$ such that $v(x)$ does not vanish on O . Then $\hat{\Omega} = \Omega$, $\hat{n}(x) = n(x)$ and $\hat{v}(x) = v(x)$ if $v(x)$ is not a gradient flow. In the case of the gradient flow there are two solutions of the inverse problem:*

$$\hat{v}(x) = v(x) \quad \text{and} \quad \hat{v}(x) = -v(x).$$

Remark 4.1 (c.f. [E4]) Suppose that the open set O where $v(x) \neq 0$ consists of several open components O_1, \dots, O_r . Suppose there exists $b_j(x) \in C^\infty(\overline{\Omega})$, $b_j(x) = 0$ on $\partial\Omega_0$, $\frac{\partial b_j}{\partial x} = v(x)$ on O_j , $b_j = 0$ in $\overline{\Omega} \setminus O_j$, $j = 1, 2, \dots, r$.

Then we have 2^r solutions of the inverse problem where each of these solutions is equal to either $\frac{\partial b_j}{\partial x}$ or to $-\frac{\partial b_j}{\partial x}$ on O_j .

5 Artificial black holes.

Let $S(x) = 0$ be a smooth closed surface in \mathbf{R}^n such that the surface $S \times \mathbf{R} \subset \mathbf{R}^{n+1}$ is a characteristic surface for the equation (2.1), i.e.

$$(5.1) \quad \sum_{j,k=1}^n g^{jk}(x) S_{x_j}(x) S_{x_k}(x) = 0 \quad \text{when} \quad S(x) = 0.$$

Let Ω_{int} be the interior of S and Ω_{ext} be the exterior of S . The domain $\Omega_{int} \times \mathbf{R}$ is called an artificial black hole if no signal emanating from it can reach $\Omega_{ext} \times \mathbf{R}$. Analogously, $\Omega_{int} \times \mathbf{R}$ is an artificial white hole if no signal from $\Omega_{ext} \times \mathbf{R}$ can penetrate the interior of $S \times \mathbf{R}$.

Let y be any point of S , i.e. $S(y) = 0$.

Lemma 5.1. *If $S \times \mathbf{R}$ is a characteristic surface then*

$$\sum_{j=1}^n g^{j0}(y) S_{x_j}(y) \neq 0.$$

Proof: Since (2.1) is hyperbolic the equation $\sum_{j,k=0}^n g^{jk}(y)\xi_j\xi_k = 0$ has two distinct real roots $\xi_0^{(1)}(\xi), \xi_0^{(2)}(\xi)$ for any $\xi = (\xi_1, \dots, \xi_n) \neq 0$. Taking $\xi = S_x(y)$ and using (5.1) we get $g^{00}(y)\xi_0^2 + 2\sum_{j=1}^n g^{0j}(y)\xi_0 S_{x_j}(y) = 0$. Therefore $\xi_0^{(1)} = 0, \xi_0^{(2)} = -2(g^{00}(y))^{-1}\sum_{j=1}^n g^{jk}(y)S_{x_j}(y) \neq 0$. \square

It follows from Lemma 5.1 that either

$$(5.2) \quad \sum_{j=1}^n g^{j0}(y)S_{x_j}(y) > 0, \quad S(y) = 0,$$

or

$$(5.3) \quad \sum_{j=1}^n g^{j0}(y)S_{x_j}(y) < 0, \quad S(y) = 0.$$

Denote by $K^+(y) \subset \mathbf{R}^{n+1}$ the half-cone

$$(5.4) \quad K^+(y) = \left\{ (\xi_0, \xi_1, \dots, \xi_n) : \sum_{j,k=0}^n g^{jk}(y)\xi_j\xi_k > 0 \right\}$$

containing $(1, 0, \dots, 0)$ and by $K^+(y)$ the dual half-cone

$$(5.5) \quad K_+(y) = \left\{ (\dot{x}_0, \dot{x}_1, \dots, \dot{x}_n) \in \mathbf{R}^{n+1} : \sum_{j,k=0}^n g_{jk}(y)\dot{x}_j\dot{x}_k > 0, \dot{x}_0 > 0 \right\}.$$

Since $K^+(y)$ and $K_+(y)$ are dual we have

$$(5.6) \quad \sum_{j=0}^n \dot{x}_j\xi_j > 0$$

for any $(\dot{x}_0, \dots, \dot{x}_n) \in K_+(y)$ and any $(\xi_0, \dots, \xi_n) \in K^+(y)$. We choose $S_x(y)$ to be the outward normal to S . Assuming (5.2) we have $(\varepsilon, S_x(y)) \in K^+(y)$ for any $\varepsilon > 0$. Using (5.6) and taking the limit when $\varepsilon \rightarrow 0$ we get that $\sum_{j=1}^n \dot{x}_j S_{x_j}(y) \geq 0$ for all $(\dot{x}_0, \dots, \dot{x}_n) \in K_+(y)$, i.e. $\overline{K}_+(y)$ is contained in the half-space $\overline{P}_+(y) = \left\{ (\alpha_0, \alpha_1, \dots, \alpha_n) : \sum_{j=1}^n \alpha_j S_{x_j}(y) \geq 0 \right\}$. In particular, $K_+(y)$ is contained in the open half-space P_+ .

A ray $x_0 = x_0(s), x = x(s), s \geq 0$, is called a forward time-like if $\left(\frac{dx_0(s)}{ds}, \frac{dx(s)}{ds} \right) \in K_+(x(s))$ for all s . It is known (c.f. [CH]) that the domain of influence of a point (y_0, y) is the closure of all forward time-like rays

starting at (y_0, y) . Therefore since $K_+(y)$ is contained in the open half-space $P_+(y)$ for all $(y_0, y) \in S \times \mathbf{R}$ we have that the domain of influence of $\Omega_{ext} \times \mathbf{R}$ is contained in $\overline{\Omega}_{ext} \times \mathbf{R}$, i.e. $\Omega_{int} \times \mathbf{R}$ is a white hole, since no signal from $\Omega_{ext} \times \mathbf{R}$ may reach $\Omega_{int} \times \mathbf{R}$.

Consider now the case when (5.3) holds. Then $(\varepsilon, -S_x(y)) \in K^+(y)$ for any $\varepsilon > 0, y \in S$. Therefore passing to the limit when $\varepsilon \rightarrow 0$ we get that $K_+(y)$ is contained in the half-space $P_-(y) = \left\{ (\alpha_0, \alpha_1, \dots, \alpha_n) : \sum_{j=1}^n \alpha_j S_{x_j}(y) < 0 \right\}$. Since $S_x(y)$ is the outward normal to $S, y \in S$, we get that the domain of influence of $\Omega_{int} \times \mathbf{R}$ is contained in $\overline{\Omega}_{int} \times \mathbf{R}$, i.e. $\Omega_{int} \times \mathbf{R}$ is a black hole. We proved the following theorem:

Theorem 5.1. *Let $S \times \mathbf{R}$ be a characteristic surface for (2.1). Then $\Omega_{int} \times \mathbf{R}$ is a white hole if (5.2) holds and a black hole if (5.3) holds.*

In §8 and §9 we will give another proof of this theorem.

Let $\Delta(x) = \det[g^{jk}(x)]_{j,k=1}^n$. We assume that the surface $S_\Delta = \{x : \Delta(x) = 0\}$ is a smooth closed surface. Let Ω_{int} be the interior of S_Δ and Ω_{ext} be the exterior of S_Δ . We assume that $\Delta(x) > 0$ in $\overline{\Omega} \cap \Omega_{ext}$ and $\Delta(x) < 0$ in $\overline{\Omega} \cap \Omega_{int}$. Borrowing the terminology from the general relativity we shall call S_Δ the ergosphere. If $S_\Delta \times \mathbf{R}$ is a characteristic surface for (2.1) then $\Omega_{int} \times \mathbf{R}$ is a black hole if (5.3) holds and a white hole if (5.2) holds. In the case of the Gordon equation the ergosphere has the form

$$S_\Delta = \left\{ x : |w(x)|^2 = \frac{c^2}{n^2} \right\},$$

and

$$g^{0j}(x) = \frac{(n^2(x) - 1)cw_j(x)}{c^2 - |w|^2}.$$

If $S_\Delta \times \mathbf{R}$ is a characteristic surface then the normal to S_Δ is colinear to $w(y)$ and $\Omega_{int} \times \mathbf{R}$ will be a black hole if $w(y)$ is pointed inside Ω_{int} , and $\Omega_{int} \times \mathbf{R}$ will be a white hole if $w(y)$ is pointed inside Ω_{ext} .

Note that the black or white holes with the boundary $S_\Delta \times \mathbf{R}$ are not stable: If we perturb slightly the metric $[g_{jk}(x)]_{j,k=0}^n$ then the ergosphere changes slightly. However it will not necessary remain a characteristic surface and the black or the white hole will disappear. In the next section we will find stable black and white holes.

6 Stable black and white holes.

Consider the case $n = 2$, i.e. the case of two space variables $x = (x_1, x_2)$. Let S_Δ be the ergosphere, i.e. $\Delta(x) = g^{11}(x)g^{22}(x) - (g^{12}(x))^2 = 0$ on S_Δ . Suppose S_Δ is a closed smooth curve and let S_1 be a smooth closed curve inside S_Δ . Denote by Ω_e the domain between S_Δ and S_1 and assume that $\Omega_e \subset \Omega$. We shall call Ω_e the ergoregion. We assume that $\Delta(x) < 0$ on $\overline{\Omega_e} \setminus S_\Delta$ and that S_Δ is not characteristic at any $y \in S_\Delta$, i.e.

$$(6.1) \quad \sum_{j,k=1}^2 g^{jk}(y) \nu_j(y) \nu_k(y) \neq 0, \quad \forall y \in S_\Delta,$$

where $(\nu_1(y), \nu_2(y))$ is the normal to S_Δ . Since $\Delta(x) < 0$ in Ω_e we can define (locally) two families of characteristic curves $S^\pm(x) = \text{const}$ satisfying

$$(6.2) \quad \sum_{j,k=1}^2 g^{jk}(x) S_{x_j}^\pm(x) S_{x_k}^\pm(x) = 0, \quad x \in \Omega_e.$$

It is shown in [E5] that there are two families $f^\pm(x)$ of vector fields such that $f^\pm(x) \neq (0, 0)$ for $\forall x \in \overline{\Omega_e}$, $f^+(x) \neq f^-(x)$ for $x \in \overline{\Omega_e} \setminus S_\Delta$, $f^+(y) = f^-(y)$ on S_Δ , and $f^\pm(x)$ are tangent to $S^\pm(x) = \text{const}$.

Consider two systems of differential equations:

$$(6.3) \quad \frac{d\hat{x}^+(\sigma)}{d\sigma} = f^+(\hat{x}^+(\sigma)), \quad \sigma \geq 0, \quad \hat{x}^+(0) = y \in S_\Delta,$$

$$(6.4) \quad \frac{d\hat{x}^-(\sigma)}{d\sigma} = f^-(\hat{x}^-(\sigma)), \quad \sigma \geq 0, \quad \hat{x}^-(0) = y \in S_\Delta,$$

Note that $x = \hat{x}^\pm(\sigma) = y, \sigma \geq 0$, are parametric equations of characteristics (6.2). It follows from (6.1) that $f^+(y) = f^-(y)$ is not tangent to S_Δ for all $y \in S_\Delta$. Since the rank of $[g^{jk}(y)]_{j,k=1}^2$ on S_Δ is 1, one can choose a smooth vector $b(y), y \in S_\Delta$, such that

$$(6.5) \quad \sum_{k=1}^2 g^{jk}(y) b_k(y) = 0, \quad j = 1, 2.$$

Note that $f^\pm(y) \cdot b(y) = 0$. We choose $f^\pm(y)$ to be pointed inside S_Δ .

Consider the equations for the null-bicharacteristics:

$$(6.6) \quad \frac{dx_j(s)}{ds} = 2 \sum_{k=0}^2 g^{jk}(x(s)) \xi_k(s), \quad x_j(0) = y_j, \quad 0 \leq j \leq 2,$$

$$(6.7) \quad \frac{d\xi_p(s)}{ds} = - \sum_{j,k=0}^2 g_{x_p}^{jk}(s) \xi_j(s) \xi_k(s), \quad \xi_p(0) = \eta_p, \quad 0 \leq p \leq 2.$$

Here $x(s) = (x_1(s), x_2(s))$. Since $g^{jk}(x)$ are independent of x_0 we have that $\xi_0(s) = \eta_s, \forall s$, and we choose $\eta_0 = 0$.

The bicharacteristic (6.6), (6.7) is a null-bicharacteristics if

$$\sum_{j,k=0}^2 g^{jk}(y) \eta_j \eta_k = 0.$$

Choosing $\eta_j = \pm b(y)$, $1 \leq j \leq 2$, $\eta_0 = 0$ we get two null-bicharacteristics $x_0 = x_0^\pm(s)$, $x = x^\pm(s)$, $\xi_0 = 0, \xi = \xi^\pm(s)$ such that the projection of these null-bicharacteristics on the (x_1, x_2) -plane coincide with solutions $x = \hat{x}^\pm(\sigma)$ of the systems (6.3), (6.4), i.e $x = \hat{x}^\pm(\sigma)$, $\sigma \geq 0$ and $x = x^\pm(s)$ are equal after a reparametrization $\sigma = \sigma^\pm(s)$, $\frac{d\sigma^\pm(s)}{ds} > 0$ for $s > 0$. We consider forward null-bicharacteristics, i.e. $\frac{dx_0^\pm(s)}{ds} > 0$ for all s . Therefore one can take the time variable x_0 as a parameter on $x = x^\pm(\sigma)$.

The key observation in [E5] is that for one of $x = \hat{x}^\pm(\sigma)$, say for $x = \hat{x}^+(\sigma)$, $\sigma = \sigma^+(s^+(x_0))$ increases when x_0 increases, and for $x = \hat{x}^-(\sigma)$, $\sigma = \sigma^-(s^-(x_0))$ decreases when x_0 increases.

Now we impose conditions on S_1 that will guarantee the existence of black and white holes in Ω_e . We assume that S_1 is not characteristic.

Let $N(y)$ be the outward unit normal to S_1 , $y \in S_1$. Suppose that either

$$(a) \quad \overline{K}_+(y) \text{ is contained in the open half-space } Q_+ = \{(\alpha_0, \alpha_1, \alpha_2) : (\alpha_0, \alpha_1, \alpha_2) \cdot (0, N(y)) > 0,$$

or

$$(b) \quad \overline{K}_+(y) \text{ is contained in the open half-space } Q_- = \{(\alpha_0, \alpha_1, \alpha_2) : (\alpha_0, \alpha_1, \alpha_2) \cdot (0, N(y)) < 0,$$

Remark 6.1 There are equivalent forms of conditions (a) and (b). Since $\sum_{j,k=0}^n g_{jk}(x(s)) \frac{dx_j(s)}{ds} \frac{dx_k(s)}{ds} = 0$ for the null-bicharacteristics we have that $\left(\frac{dx_0(s)}{ds}, \frac{dx_1(s)}{ds}, \frac{dx_2(s)}{ds} \right) \in \overline{K}_+(y)$ for the forward null-bicharacteristic when $x(s_1) = y \in S_1$. Therefore the condition (a) is equivalent to the condition:

- (a₁) The projection on (x_1, x_2) -plane of all forward null-characteristics passing through $y \in S_1$ leave Ω_e when x_0 increases.

Further, the condition (a₁) is equivalent to the following more simple condition:

Let $x = x^\pm(s)$ be the projection on (x_1, x_2) -plane of two forward null-bicharacteristics such that $x = x^\pm(s)$ are the parametric equations of the characteristics $S^\pm(x) = \text{const}$, i.e. $x = x^\pm(s)$ are solutions of the differential equations (6.3), (6.4) after a reparametrization. Assume that

- (a₂) $\frac{dx^\pm(s_1)}{ds} \cdot N(y) > 0$ when $x^\pm(s_1) = y$.

The condition (a₂) follows from (a₁). The inverse is also true since the set of directions of the projections of all forward null-bicharacteristics passing through y is bounded by $\frac{dx^+(s_1)}{ds}$ and $\frac{dx^-(s_1)}{ds}$.

Conditions (b₁), (b₂) are similar to (a₁), (a₂) when the sign of the inner product in (a) is negative.

Theorem 6.1. (c.f. [E5]) Let $\partial\Omega_e = S_\Delta \cup S_1$, where S_Δ is the ergosphere, i.e. $\Delta(y) = 0$ on S_Δ . Suppose (6.1) holds on S_Δ and either (a) or (b) hold on S_1 . Then there exists a closed Jordan curve $S_0(x) = 0$ inside Δ_e such that $S_0 \times \mathbf{R}$ is the boundary of either black or white hole.

The proof of Theorem 6.1 is based on the Poincare-Bendixson theorem (c.f. [H]). Suppose (a) holds. Then the solution of (6.4) cannot reach S_1 . Indeed, suppose $\hat{x}^-(\sigma_1) = y_1 \in S_1$ for some $\sigma_1 > 0$. Then $\hat{x}^-(\sigma)$ leaves Ω_e when $\sigma > \sigma_1$. From other side, when σ increases x_0 decreases. Therefore $x = \hat{x}^-(\sigma^-(x_0))$ leaves Ω_e when x_0 decreases, and this contradicts the condition (a₁). Since $x = \hat{x}^-(\sigma)$ never reaches S_1 the limit set of the trajectory $x = \hat{x}^-(\sigma)$ is contained inside Ω_e . Then by the Poincare-Bendixson theorem there exists a limit cycle $S_0(x) = 0$, i.e. a Jordan curve that is a periodic solution of $\frac{d\hat{x}}{d\sigma} = f^-(\hat{x}(\sigma))$. Therefore $S_0 \times \mathbf{R}$ is a characteristic surface and it is the boundary of a black or a white hole. In the case when the condition (b) holds we have that the solution of (6.3) never reach S_1 . Therefore again by the Poincare-Bendixson Theorem there exists a black or white hole.

Applying Theorem 6.1 to the Gordon equation we get

Theorem 6.2. *Let S_Δ be the ergosphere, i.e. $|w|^2 = \frac{c^2}{n^2(x)}$. Suppose $w(x)$ is not colinear with the normal to S_Δ for any $x \in S_\Delta$. Suppose that either*

$$(6.8) \quad (n^2(x) - 1)^{\frac{1}{2}}(v(x) \cdot N(x)) > 1 \quad \text{on } S_1,$$

or

$$(6.9) \quad (n^2(x) - 1)^{\frac{1}{2}}(v(x) \cdot N(x)) < -1 \quad \text{on } S_1,$$

where $v(x) = \left(1 - \frac{|w|^2}{c^2}\right)^{-\frac{1}{2}} \frac{w(x)}{c}$, $N(x)$ is the outward unit normal to S_1 .

Then there exists a limit cycle $S_0(x) = 0$ and $S_0 \times \mathbf{R}$ is the boundary of a black or a white hole.

Remark 6.1 Note that the black or white holes obtained by Theorems 6.1 and 6.2 are stable since the assumptions remain valid when we slightly deform the metric.

7 Rotating black holes. Examples.

Example 1 ([V]). Acoustic black hole. Consider a fluid flow with velocity field

$$(7.1) \quad v = (v^1, v^2) = \frac{A}{r} \hat{r} + \frac{B}{r} \hat{\theta},$$

where $r = |x|$, $\hat{r} = \left(\frac{x_1}{|x|}, \frac{x_2}{|x|}\right)$, $\hat{\theta} = \left(-\frac{x_2}{|x|}, \frac{x_1}{|x|}\right)$, A and B are constants. The inverse of the metric tensor has the following form in this case:

$$(7.2) \quad \begin{aligned} g^{00} &= \frac{1}{\rho c}, \quad g^{oj} = g^{j0} = \frac{1}{\rho c} v^j, \quad 1 \leq j \leq 2, \\ g^{jk} &= \frac{1}{\rho c} (-c^2 \delta_{ij} + v^j v^k), \quad 1 \leq j, k \leq 2, \end{aligned}$$

where c is the sound speed, ρ is the density.

Consider the case $A > 0$, $B > 0$. Assume $\rho = c = 1$. Then the ergosphere is $r = \sqrt{A^2 + B^2}$. Consider the domain $\Omega_e = \{r_1 \leq r \leq \sqrt{A^2 + B^2}\}$, where $r_1 < A$. In polar coordinates (r, θ) the differential equations (6.3), (6.4) have the form:

$$(7.3) \quad \frac{dr}{ds} = A^2 - r^2, \quad \frac{d\theta}{ds} = \frac{AB}{r} + \sqrt{A^2 + B^2 - r^2},$$

and

$$(7.4) \quad \frac{dr}{ds} = -1, \quad \frac{d\theta}{ds} = \frac{1 - \frac{B^2}{r^2}}{\frac{AB}{r} + \sqrt{A^2 + B^2 - r^2}}.$$

We have that $r = A$ is a limit cycle and $\{r = A\} \times \mathbf{R}$ is the boundary of a white hole (c.f. [E5]).

Example 2 Consider a fluid flow with the velocity

$$v = A(r)\hat{r} + B(r)\hat{\theta},$$

where $r_1 \leq r \leq r_0$, $A(r), B(r)$ are smooth, $B(r) > 0$, $A^2(r_0) + B^2(r_0) = 1$, $A^2(r) + B^2(r) > 1$ on $[r_1, r_0)$, $A(r) + 1$ has simple zeros $\alpha_1, \dots, \alpha_{m_1}$ on (r_1, r_0) , $A(r) - 1$ has simple zeros $\beta_1, \dots, \beta_{m_2}$ on (r_1, r_0) , $\alpha_j \neq \beta_k$, $\forall j, \forall k$, $|A(r_1)| > 1$. Here $r = r_0$ is the ergosphere. The differential equations (6.3), (6.4) have the following form in polar coordinates (r, θ) :

$$(7.5) \quad \frac{dr}{ds} = A(r) - 1, \quad \frac{d\theta}{ds} = \frac{A(r)B(r) + \sqrt{A^2(r) + B^2(r) - 1}}{A(r) + 1},$$

and

$$(7.6) \quad \frac{dr}{ds} = A(r) + 1, \quad \frac{d\theta}{ds} = \frac{A(r)B(r) - \sqrt{A^2(r) + B^2(r) - 1}}{A(r) - 1}.$$

Here $r = \alpha_j$, $1 \leq j \leq m_1$, and $r = \beta_k$, $1 \leq k \leq m_2$ are limit cycles and there are $m_1 + m_2$ black and white holes.

Axially symmetric metrics.

Consider the equation (2.1) in $\Omega \times \mathbf{R}$ where Ω is a three-dimensional domain. Let (r, θ, φ) be the spherical coordinates in \mathbf{R}^3 . Suppose g^{jk} are independent of φ .

Consider a characteristic surface S independent of φ and x_0 , i.e. S depends on r and θ only. Then S satisfies an equation

$$(7.7) \quad a^{11}(r, \theta) \left(\frac{\partial S}{\partial r} \right)^2 + 2a^{12}(r, \theta) \frac{\partial S}{\partial r} \frac{\partial S}{\partial \theta} + a^{22}(r, \theta) \left(\frac{\partial S}{\partial \theta} \right)^2 = 0.$$

We assume that $a^{ij}(r, \theta)$ are also independent of φ , $1 \leq j, k \leq 2$.

Consider (7.7) in two-dimensional domain ω where $\delta_1 \leq r \leq \delta_2$, $0 < \delta_3 < \theta < \pi - \delta_4$ when $(r, \theta) \in \omega$. Here $\delta_j > 0$, $1 \leq j \leq 4$.

Assuming that ω and $a^{jk}(r, \theta)$, $1 \leq j, k \leq 2$, satisfy the condition of Theorem 6.1, we can prove the existence of black or white holes whose boundary is $S_0 \times S^1 \times \mathbf{R}$, where $\varphi \in S^1$, $x_0 \in \mathbf{R}$ and S_0 is a Jordan curve in ω .

Such black (white) holes are called the rotating black (white) holes.

8 Black holes and inverse problems I.

In this section we consider the case of the black or the white hole bounded by $S_\Delta \times \mathbf{R}$, where S_Δ is the ergosphere. Suppose Ω_{int} is a black hole, i.e. (5.3) holds. Let L be the operator (2.1). Consider $u(x_0, x)$ in $(\Omega \cap \Omega_{ext}) \times (0, T)$ such that

$$(8.1) \quad Lu = f, \quad (x_0, x) \in (\Omega \cap \Omega_{ext}) \times (0, T),$$

$$(8.2) \quad u(0, x) = 0, \quad \frac{\partial u(0, x)}{\partial x_0} = 0, \quad x \in (\Omega \cap \Omega_{ext}),$$

$$(8.3) \quad u|_{\partial\Omega_0 \times (0, T)} = g.$$

We do not impose any boundary conditions on $S_\Delta \times (0, T)$ and we assume, for the simplicity, that there is no obstacles between $\partial\Omega_0$ and S_Δ . We shall prove an estimate of $u(x_0, x)$ in terms of g and f .

Denote $Hu = \sum_{j=1}^n g^{0j}(x)u_{x_j}$. Consider the equality

$$(Lu, g^{00}u_{x_0} + Hu) = (f, g^{00}u_{x_0} + Hu),$$

where (u, v) is the inner product in $L_2((\Omega \cap \Omega_{int}) \times (0, T))$. We shall denote by $Q_p(u, v)$ for $p \geq 1$ the expression of the form:

$$(8.4) \quad (Q_p u, v) = \int_0^T \int_{\Omega \cap \Omega_{int}} \sum_{j,k=0}^n q_{jkp}(x) u_{x_j} v_{x_k} dx dx_0.$$

Denote

$$(8.5) \quad I_1 = \left(\sum_{j=1}^n \frac{1}{\sqrt{|g|}} \frac{\partial}{\partial x_j} \left(\sqrt{|g|} g^{0j} \frac{\partial u}{\partial x_0} \right) + \sum_{j=0}^n \frac{1}{\sqrt{|g|}} \frac{\partial}{\partial x_0} \left(\sqrt{|g|} g^{0j} \frac{\partial u}{\partial x_j} \right), g^{00}u_{x_0} + Hu \right) \\ \stackrel{def}{=} I_{11} + I_{12} + I_{13}$$

We have

$$(8.6) \quad I_{11} = \left(\sum_{j=0}^n \frac{1}{\sqrt{|g|}} \frac{\partial}{\partial x_0} \left(\sqrt{|g|} g^{0j} \frac{\partial u}{\partial x_j} \right), g^{00}u_{x_0} + Hu \right) \\ = \frac{1}{2} \int_{\Omega \cap \Omega_{int}} \left(\sum_{j=0}^n g^{0j} u_{x_j} \right)^2 dx \Big|_0^T + Q_1(u, u),$$

where $a|_0^T$ means $a(T) - a(0)$. Note that

$$(8.7) \quad \begin{aligned} I_{12} &= \left(\sum_{j=1}^n \frac{1}{\sqrt{|g|}} \frac{\partial}{\partial x_j} \left(\sqrt{|g|} g^{0j} \frac{\partial u}{\partial x_0} \right), Hu \right) \\ &= \frac{1}{2} \int_{\Omega \cup \Omega_{ext}} (Hu)^2 dx \Big|_0^T + Q_2(u, u). \end{aligned}$$

Also we have

$$(8.8) \quad \begin{aligned} I_{13} &= \left(\frac{1}{\sqrt{|g|}} \sum_{j=1}^n \frac{\partial}{\partial x_j} \left(\sqrt{|g|} g^{0j} \frac{\partial u}{\partial x_0} \right), g^{00} u_{x_0} \right) \\ &= \frac{1}{2} \int_{\Omega \cup \Omega_{ext}} \sum_{j=1}^n \frac{\partial}{\partial x_j} (g^{0j} g^{00} u_{x_0}^2) dx dx_0 + Q_3(u, u). \end{aligned}$$

By the divergence theorem we get

$$(8.9) \quad \begin{aligned} I_{13} &= - \int_0^T \int_{S_\Delta} \frac{1}{2} \left(\sum_{j=1}^n g^{0j}(x) \nu_j(x) \right) g^{00} u_{x_0}^2 ds dx_0 \\ &\quad + \int_0^T \int_{\partial\Omega_0} \frac{1}{2} \left(\sum_{j=1}^n g^{0j}(x) N_j(x) \right) g^{00} u_{x_0}^2 ds dx_0 + Q_3(u, u), \end{aligned}$$

where ds is the area element on S_Δ and $\partial\Omega_0$, respectively, $N(x) = (N_1, \dots, N_n)$ is the outward unit normal to $\partial\Omega_0$ and $\nu = (\nu_1, \dots, \nu_n)$ is an outward normal to S_Δ . Note that ν is the inward normal with respect to $\Omega \cap \Omega_{ext}$. Therefore

$$(8.10) \quad \begin{aligned} I_1 &= I_{11} + I_{12} + I_{13} \\ &= \frac{1}{2} \int_{\Omega \cap \Omega_{ext}} \left[\left(\sum_{j=0}^n g^j u_{x_j} \right)^2 + (Hu)^2 \right] dx \Big|_0^T \\ &\quad - \int_0^T \int_{S_\Delta} \frac{1}{2} \left(\sum_{j=1}^n g^{0j}(x) \nu_j(x) \right) g^{00} u_{x_0}^2 ds dx_0 \\ &\quad + \int_0^T \int_{\partial\Omega_0} \frac{1}{2} \left(\sum_{j=1}^n g^{0j}(x) N_j(x) \right) g^{00} u_{x_0}^2 ds dx_0 + Q_4(u, u). \end{aligned}$$

Now consider

$$I_2 = \left(\sum_{j,k=1}^n \frac{1}{\sqrt{|g|}} \frac{\partial}{\partial x_j} \left(\sqrt{|g|} g^{jk} \frac{\partial u}{\partial x_k} \right), g^{00} u_{x_0} \right).$$

Integrating by parts in x_j and taking into account that

$$-\sum g^{jk} u_{x_k} u_{x_0 x_j} = -\frac{1}{2} \frac{\partial}{\partial x_0} \left(\sum_{j,k=1}^n g^{jk} u_{x_j} u_{x_k} \right),$$

we get:

$$(8.11) \quad \begin{aligned} I_2 = & -\frac{1}{2} \int_{\Omega \cap \Omega_{ext}} g^{00} \sum_{j,k=1}^n g^{jk}(x) u_{x_j} u_{x_k} dx \Big|_0^T \\ & - \int_0^T \int_{S_\Delta} \left(\sum_{j,k=1}^n g^{jk} u_{x_j}(x) \nu_k(x) g^{00} u_{x_0} \right) ds dx_0 \\ & + \int_0^T \int_{\partial \Omega_0} \left(\sum_{j,k=1}^n g^{jk}(x) u_{x_j} N_k(x) g^{00} u_{x_0} \right) ds dx_0 + Q_5(u, u). \end{aligned}$$

Since S_Δ is an ergosphere and a characteristic surface we have (c.f. (6.5)):

$$(8.12) \quad \sum_{k=1}^n g^{jk}(x) \nu_k(x) = 0 \quad \text{on } S_\Delta, \quad j = 1, \dots, n.$$

Let

$$(8.13) \quad I_3 = \left(\sum_{j,k=1}^n \frac{\partial}{\partial x_j} \left(\sqrt{|g|} g^{jk} \frac{\partial u}{\partial x_k} \right), Hu \right).$$

Integrating by parts in x_j , $1 \leq j \leq n$, we get

$$(8.14) \quad \begin{aligned} I_3 = & - \left(\sum_{j,k=1}^n g^{jk} u_{x_k}, Hu_{x_j} \right) - \int_0^T \int_{S_\Delta} \sum_{j,k=1}^n g^{jk} \nu_j u_{x_k} Hu \, ds dx_0 \\ & + \int_0^T \int_{\partial \Omega_0} \sum_{j,k=1}^n g^{jk} N_j u_{x_k} Hu \, ds dx_0 + Q_6(u, u). \end{aligned}$$

We have

$$(8.15) \quad \begin{aligned} & \int_0^T \int_{\Omega \cap \Omega_{ext}} \frac{\partial}{\partial x_p} g^{0p} \sum_{j,k=1}^n g^{jk} u_{x_j} u_{x_k} dx dx_0 \\ = & - \int_0^T \int_{S_\Delta} g^{0p} \nu_p \sum_{j,k=1}^n g^{jk} u_{x_j} u_{x_k} ds dx_0 + \int_0^T \int_{\partial \Omega_0} g^{0p} N_p \sum_{j,k=1}^n g^{jk} u_{x_j} u_{x_k} ds dx_0 \end{aligned}$$

Using (8.15) we obtain

$$(8.16) \quad - \left(\sum_{j,k=1}^n g^{jk} u_{x_k}, H u_{x_j} \right) = \int_0^T \int_{S_\Delta} \left(\sum_{p=1}^n g^{0p} \nu_p \right) \sum_{j,k=1}^n g^{jk} u_{x_j} u_{x_k} ds dx_0 \\ - \int_0^T \int_{\partial\Omega_0} \left(\sum_{p=1}^n g^{0p} N_p \right) \sum_{j,k=1}^n g^{jk} u_{x_j} u_{x_k} ds dx_0 + Q_7(u, u).$$

Note that the first integral in (8.16) is nonnegative since $\sum_{p=1}^n g^{0p} \nu_p < 0$ on S_Δ and $\sum_{j,k=1}^n g^{jk} u_{x_j} u_{x_k} \leq 0$ in $\Omega \cap \bar{\Omega}_{int}$ since the matrix $[g^{jk}]_{j,k=1}^n$ has one zero eigenvalue and $n-1$ negative eigenvalues on S_Δ .

Now we shall estimate the integrals over $\partial\Omega_0 \times (0, T)$. Let $\alpha(x) \in C_0^\infty(\mathbf{R}^n)$, $\alpha(x) = 1$ near $\partial\Omega_0$ and $\alpha(x) = 0$ near S_Δ . We have from (8.1), (8.2), (8.3) that $v = \alpha u$ satisfies

$$(8.17) \quad Lv = \alpha f + L_1 u \quad \text{in } \Omega_0 \times (0, T),$$

$$(8.18) \quad v|_{x_0=0} = 0, \quad v_{x_0}|_{x_0=0} = 0, \quad x \in \Omega_0,$$

$$(8.19) \quad v|_{\partial\Omega_0 \times (0, T)} = g.$$

Here g is the same as in (8.3) and $\text{ord } L_1 \leq 1$.

Since L is strictly hyperbolic and $\partial\Omega_0$ is not characteristic the following estimate for the solution of (8.17), (8.18), (8.19) holds (c.f. for example, [Ho], see also [E7]):

$$(8.20) \quad \|v_{x_0}(T, \cdot)\|_0^2 + \|v(T, \cdot)\|_1^2 + \left[\frac{\partial v}{\partial N} \right]_0^2 \\ \leq C_T \left([g]_1^2 + \int_0^T \|f(x_0, \cdot)\|_0^2 dx_0 + \int_0^T (\|u(x_0, \cdot)\|_1^2 + \|u_{x_0}(x_0, \cdot)\|_0^2) dx_0 \right),$$

where $[w]_m$ is the norm in $H^m(\partial\Omega_0 \times (0, T))$ and $\|w(x_0, \cdot)\|_m$ is the norm in $H^m(\Omega_0)$, x_0 is fixed. All integrals over $\partial\Omega_0 \times (0, T)$ in I_2 and I_3 have the form

$$I_4 = \int_0^T \int_{\partial\Omega_0} \sum_{j,k=0}^n p_{jk} u_{x_j} u_{x_k} ds dx_0.$$

Therefore

$$\begin{aligned}
(8.21) \quad |I_4| &\leq C \left([g]_1^2 + \left[\frac{\partial v}{\partial N} \right]_0^2 \right) \\
&\leq C \left([g]_1^2 + \int_0^T \|f(x_0, \cdot)\|_0^2 dx_0 + \int_0^T (\|u(x_0, \cdot)\|_1^2 + \|u_{x_0}(x_0, \cdot)\|_0^2) dx_0 \right),
\end{aligned}$$

where we used (8.20) to estimate $\left[\frac{\partial v}{\partial N} \right]_0^2$. Note that the norm

$$(8.22) \quad \int_{\Omega \cap \Omega_{ext}} [(u_{x_0} + Hu)^2 + (Hu)^2 - \sum_{j,k=1}^n g^{jk} u_{x_j} u_{x_k}] dx$$

is equivalent to $\|u(x_0, \cdot)\|_1^2 + \|u_{x_0}(x_0, \cdot)\|_0^2$. Also

$$(8.23) \quad |(f, g^{00} u_{x_0} + Hu)| \leq \frac{1}{2} \int_0^T \|f(y_0, \cdot)\|_0^2 dy_0 + C \int_0^T (\|u_{x_0}\|_0^2 + \|u(y_0, \cdot)\|_1^2) dy_0.$$

Combining (8.10), (8.11), (8.14), (8.16), (8.21), taking into account (8.12) and applying all estimates to the interval $(0, t_0)$ instead of $(0, T)$, $t_0 \leq T$, we get

$$\begin{aligned}
(8.24) \quad & C(\|u(t_0, \cdot)\|_1^2 + \|u_{x_0}(t_0, \cdot)\|_0^2) \\
& - \int_0^T \int_{S_\Delta} \left(\sum_{j=1}^n g^{0j} \nu_j(x) \right) \left(g^{00} u_{x_0}^2 - \sum_{j,k=1}^n g^{jk} u_{x_j} u_{x_k} \right) ds dx_0 \\
& \leq C \int_0^T \|f(y_0, \cdot)\|_0^2 dy_0 + C \int_0^{t_0} (\|u(y_0, \cdot)\|_1^2 + \|u_{x_0}(y_0, \cdot)\|_0^2) dy_0 + C[g]_1^2.
\end{aligned}$$

Note that the inequality $b(t) \leq C \int_0^t b(\tau) d\tau + d$ implies $b(t) \leq e^{ct} d$. Therefore we get

$$\begin{aligned}
(8.25) \quad & C(\|u(t_0, \cdot)\|_1^2 + \|u_{x_0}(t_0, \cdot)\|_0^2) \\
& - \int_0^T \int_{S_\Delta} \left(\sum_{j=1}^n g^{0j} \nu_j(x) \right) \left(g^{00} u_{x_0}^2 - \sum_{j,k=1}^n g^{jk} u_{x_j} u_{x_k} \right) ds dx_0 \\
& \leq C \int_0^T \|f(y_0, \cdot)\|_0^2 dy_0 + C[g]_1^2,
\end{aligned}$$

where $0 \leq t_0 \leq T$.

We proved the following theorem:

Theorem 8.1. *Let $u(x_0, x)$ be the solution of (8.1), (8.2), (8.3) in $\Omega \cap \Omega_{ext}$. Let the ergosphere S_Δ be a characteristic surface and (5.3) holds. Then $u(x_0, x)$ satisfies (8.25).*

Note that $g^{00}u_{x_0}^2 - \sum_{j,k=1}^n u_{x_j}u_{x_k} \geq 0$ on $S_\Delta \times [0, T]$ since S_Δ is an ergosphere.

Theorem 8.1 implies that the domain of dependence of $(\Omega \cap \Omega_{ext}) \times \mathbf{R}$ is contained in $(\overline{\Omega} \cap \overline{\Omega}_{ext}) \times \mathbf{R}$. Suppose u is a solution of (2.1) and $\text{supp } u \subset \overline{\Omega}_{int}$ for $x_0 \leq t_0$, i.e. $u = 0$ for $x \in (\Omega_{ext} \cap \Omega) \times (-\infty, t_0)$. Then Theorem 8.1 implies that $u = 0$ for $(\overline{\Omega} \cap \overline{\Omega}_{ext}) \times [t_0, +\infty)$, i.e. $\text{supp } u \subset \overline{\Omega}_{int} \times \mathbf{R}$. Therefore the domain of influence of $\Omega_{int} \times \mathbf{R}$ is contained in $\overline{\Omega}_{int} \times \mathbf{R}$, i.e. is a black hole.

Now we shall discuss the nonuniqueness of the inverse problem in the presence of a black hole. Consider two initial-boundary value problem (2.1), (2.5), (2.6) for the operators L_1 and L_2 that differ only in Ω_{int} . Since $L_1 = L_2$ in Ω_{ext} and we assume f is the same for L_1 and L_2 we get by Theorem 8.1 that $u_1 = u_2$ in $(\Omega \cap \Omega_{ext}) \cap \mathbf{R}$ where u_1 and u_2 are the solutions of the corresponding initial-boundary value problems. Therefore $\Lambda_1 = \Lambda_2$ on $\partial\Omega_0 \times \mathbf{R}$. Since $L_1 \neq L_2$ in Ω_{int} , i.e. we have a nonuniqueness of the inverse problem. \square

Consider now the case when S_Δ is a characteristic surface and (5.2) holds. Suppose $\Omega_{int} \cap \Omega$ contains an obstacle Ω_1 (it may be no obstacles or more than one obstacle, but we consider the case of one obstacle for the definiteness). Integrating by parts as in the proof of Theorem 8.1 and using (5.2) instead of (5.3) we get the following theorem:

Theorem 8.2. *Consider the initial-boundary value problem:*

$$(8.26) \quad Lu = f \quad \text{in} \quad (\Omega_{int} \cap \Omega) \times (0, T),$$

$$(8.27) \quad u|_{\partial\Omega_1 \times (0, T)} = g, \quad u(0, x) = 0, \quad u_{x_0}(0, x) = 0 \quad \text{in} \quad \Omega \cap \Omega_{int}.$$

Suppose that the ergosphere S_Δ is a characteristic surface and (5.2) holds. Then an estimate of the form (8.25) holds in $(\Omega_{int} \cap \Omega) \times (0, T)$ with the following changes: Integral over $S_\Delta \times (0, T)$ must be taken with plus sign, $\|u\|_s$ are the norms in $H^s(\Omega_{int} \cap \Omega)$, $[g]_1$ is the norm in $H^1(\partial\Omega_1 \times (0, T))$.

The consequence of Theorem 8.2 is that the domain of dependence of $(\Omega_{int} \cap \Omega) \times \mathbf{R}$ is contained in $(\overline{\Omega}_{int} \cap \overline{\Omega}) \times \mathbf{R}$. Therefore if $u(x_0, x)$ is the

solution of (2.1) and $\text{supp } u \subset \overline{\Omega}_{ext} \cap \overline{\Omega}$ for $x_0 \leq t_0$ then $\text{supp } u \subset \overline{\Omega}_{ext} \cap \overline{\Omega}$ for all $x_0 > t_0$, i.e. $\Omega_{int} \times \mathbf{R}$ is a white hole. If $u(x_0, x)$ is the solution of (2.1), (2.5), (2.6), then $u(0, x) = 0$ in $\Omega_{int} \cap \Omega$, $u|_{\partial\Omega_1 \times \mathbf{R}} = 0$. Then by Theorem 8.2 $u = 0$ in $(\Omega_{int} \cap \Omega) \times \mathbf{R}$. Therefore we can change the coefficients of L in $\Omega_{int} \cap \Omega$ without changing the solution of (2.1), (2.5), (2.6), i.e. in the case of a white hole we again have a nonuniqueness of the solution of the inverse problem. \square

Let $u(x_0, x)$ be the solution of

$$(8.28) \quad Lu = f \quad \text{in } \Omega_{ext} \times (0, T),$$

$$(8.29) \quad u(0, x) = \varphi_0(x), \quad u_{x_0}(0, x) = \varphi_1(x), \quad x \in \Omega_{ext},$$

i.e. we consider (8.28), (8.29) in unbounded domain $\Omega_{ext} = \mathbf{R}^n \setminus \overline{\Omega}_{int}$. We assume that $[g^{jk}(x)]_{j,k=1}^n$ are smooth and have bounded derivatives of any order, $g^{00}(x) \geq C_0 > 0$, $\sum_{j,k=1}^n g^{jk}(x) \xi_j \xi_k \leq -C_0 \sum_{j=1}^n \xi_j^2$, $x \in \overline{\Omega}_{ext}$ is large, and (5.3) holds. Repeating the proof of Theorem 8.1 (with the simplification that we do not have the boundary condition (8.3)), we get for any $T > 0$:

$$(8.30) \quad \begin{aligned} & \max_{0 \leq x_0 \leq T} (\|u(x_0, \cdot)\|_{1, \Omega_{ext}}^2 + \|u_{x_0}(x_0, \cdot)\|_{0, \Omega_{ext}}^2) \\ & - \int_0^T \int_{S_\Delta} (u_{x_0}^2 - \sum_{j,k=1}^n g^{jk} u_{x_j} u_{x_k}) (\sum_{j=1}^n g^{j0} \nu_j) ds dy_0 \\ & \leq C([\varphi_0]_{1, \Omega_{ext}}^2 + [\varphi_1]_{0, \Omega_{ext}}^2) + C \int_0^T \|f(x_0, \cdot)\|_{0, \Omega_{ext}}^2 dx_0. \end{aligned}$$

Therefore the following theorem holds:

Theorem 8.3. *Suppose $u(x_0, x)$ satisfies (8.28), (8.29). Suppose the ergosphere S_Δ is a characteristic surface and (5.3) holds. Then the estimate (8.30) holds for any $T > 0$.*

A consequence of Theorem 8.3 is that $D_+(\Omega_{int} \times \mathbf{R}) \subset \overline{\Omega}_{int} \times \mathbf{R}$, i.e. $\Omega_{int} \times \mathbf{R}$ is a black hole. \square

An important problem is the determination of black or white holes by the boundary measurements on $\partial\Omega_0 \times (0, T)$ or on $\Gamma \times (0, T_0)$ where Γ is an open part of $\partial\Omega_0$. Let S_Δ be the ergosphere inside Ω_0 . It does not matter in this subsection whether S_Δ forms a black (or a white) hole. The following theorem is straightforward application of the proof of Theorem 2.1:

Theorem 8.4. *The DN operator Λ given on $\Gamma \times (0, +\infty)$ determines S_Δ up to a diffeomorphism (2.8).*

We note that the determination of S_Δ requires to take measurements on $\Gamma \times (0, +\infty)$. It is not enough to know the Cauchy data on $\Gamma \times (0, T)$ for any finite T . The explanation of this phenomenon is the following: The proof of Theorem 2.1 allows to recover metric tensor $[g^{jk}]$ (up to a diffeomorphism) gradually starting from the boundary $\partial\Omega_0$. The recovery of the metric at some point $x^{(1)}$ inside Ω_0 requires some observation time T_1 . When $x^{(1)}$ is deeper inside Ω_0 the observation time increases. When the point $x^{(1)}$ approaches S_Δ , i.e. $g_{00}(x^{(1)}) \rightarrow 0$, the needed observation time tends to infinity. One can see this from the fact that either forward time-like ray or backward time-like ray tends to infinity in x_0 when $x^{(1)} \rightarrow S_\Delta$ (c.f. [ER]).

9 Black holes and inverse problems II.

In this section we consider black or white holes inside the ergosphere.

Suppose $S_0 \times \mathbf{R}$ is a characteristic surface, $n \geq 2$, and $S_0 \subset S_\Delta$ where S_Δ is the ergosphere. Suppose the condition (5.3) on S_0 holds. Consider $v(x_0, x)$ in $\Omega_{ext} \times (0, T)$ such that

$$(9.1) \quad Lv = f, \quad x \in \Omega_{ext} \times (0, T),$$

$$(9.2) \quad v(0, x) = \varphi_0(x), \quad v_{x_0}(0, x) = \varphi_1(x), \quad x \in \Omega_{ext}.$$

We want to get an estimate of $v(x_0, x)$ in $\Omega_{ext} \times (0, T)$ in terms of φ_0, φ_1, f . Let $\hat{\varphi}_0, \hat{\varphi}_1, \hat{f}$ be smooth extensions of φ_0, φ_1 to \mathbf{R}^n and of f to $\mathbf{R}^n \times (0, T)$ such that

$$(9.3) \quad \hat{E}(\hat{\varphi}_0, \hat{\varphi}_1) \leq 2E(\varphi_0, \varphi_1), \quad \|\hat{f}\|_{0, \mathbf{R}^n \times (0, T)} \leq 2\|f\|_{0, \Omega_{ext} \times (0, T)},$$

where $E(\varphi_0, \varphi_1) = \int_{\Omega_{ext}} (\sum_{j=1}^n \varphi_{0x_j}^2 + \varphi_1^2) dx$ and $\tilde{E}(\tilde{\varphi}_0, \tilde{\varphi}_1)$ is a similar integral over \mathbf{R}^n . Since L is strictly hyperbolic there exists \hat{u} in $\mathbf{R}^n \times (0, T)$ such that

$$(9.4) \quad \begin{aligned} L\hat{u} &= \hat{f} \quad \text{in } \mathbf{R}^n \times (0, T), \\ \hat{u}(0, x) &= \hat{\varphi}_0(x), \quad \hat{u}_{x_0}(0, x) = \hat{\varphi}_1(x), \quad x \in \mathbf{R}^n, \end{aligned}$$

and

$$(9.5) \quad \begin{aligned} & \max_{0 \leq x_0 \leq T} (\|\hat{u}(x_0, \cdot)\|_{1, \mathbf{R}^n}^2 + \|\hat{u}_{x_0}(x_0, \cdot)\|_{0, \mathbf{R}^n}^2) \\ & \leq C \hat{E}(\hat{\varphi}_0, \hat{\varphi}_1) + C \int_0^T \int_{\mathbf{R}^n} |\hat{f}|^2 dx dx_0. \end{aligned}$$

Replacing $v = u + \hat{u}$ we get that $u(x_0, x)$ satisfies

$$(9.6) \quad \begin{aligned} & Lu = 0 \quad \text{in } \Omega_{ext} \times (0, T), \\ & u(0, x) = u_{x_0}(0, x) = 0, \quad x \in \Omega_{ext}. \end{aligned}$$

Therefore it remains to show that if $u(x_0, x)$ satisfies (9.6) then $u(x_0, x) = 0$ in $\Omega_{ext} \times (0, T)$. Note that we could use the same approach in §8 too.

Let T be small. Denote by Γ_1 the characteristic surface different from $S_0 \times \mathbf{R}$ and passing through $S_0 \times \{x_0 = T\}$. Let D_T be the domain bounded by Γ_1 , $\Gamma_2 = S_0 \times [-\varepsilon, T]$ and $\Gamma_3 = \{x_0 = -\varepsilon\}$. For arbitrary point $x^{(0)} \in S_0$ denote by D_{0T} the intersection of D_T with $\Sigma(x^{(0)}) = \{(x_0, x) : |x - x^{(0)} - (x - x^{(0)}) \cdot \nu(x^{(0)})\nu(x^{(0)})| < \varepsilon\}$, where $\nu(x^{(0)})$ is the unit outward normal to S_0 at $x^{(0)}$. Let $\alpha_j(x_0, x) \in C^\infty(D_T)$, $\sum_{j=1}^N \alpha_j \equiv 1$ in D_T , $\text{supp } \alpha_j \subset D_{jT}$, where D_{jT} corresponds to $x^{(j)} \in S_0$ instead of $x^{(0)}$, $\alpha_j = 0$ in a neighborhood of the boundary of $\Sigma(x^{(0)})$.

Let α_0 be any of α_j , $1 \leq j \leq N$. Denote $u_0 = \alpha_0 u$. Then

$$(9.7) \quad Lu_0 = f_0, \quad u_0(0, x) = u_{x_0}(0, x) = 0,$$

where $f_0 = L'u$, ord $L' \leq 1$, $\text{supp } f_0 \subset D_{0T}$. We introduce local coordinates in a neighborhood $B_{\varepsilon, T} = \{(x_0, x) : x_0 \in [-\varepsilon, T], |x - x^{(0)}| < 2\varepsilon\}$. Let $s = \varphi(x)$ be the solution of the eiconal equation

$$(9.8) \quad \sum_{j,k=1}^n g^{jk}(x) \varphi_{x_j} \varphi_{x_k} = 0 \quad \text{in } B_{\varepsilon, T}.$$

Since S_0 is inside the ergosphere, $\varphi(x)$ exists when ε and T are small. We choose $s = \varphi(x)$ such that $\varphi(x) = 0$ is the equation of S_0 near $x^{(0)}$.

Let $\tau = \psi(x_0, x)$ be the solution of the following eiconal equation:

$$(9.9) \quad \sum_{j,k=0}^n g^{jk}(x) \psi_{x_j} \psi_{x_k} = 0$$

with the initial data

$$(9.10) \quad \psi(x_0, x)|_{\Gamma_2} = T - x_0.$$

Finally denote by $y_j = \varphi_j(x_0, x)$, $1 \leq j \leq n-1$, the solution of the equation

$$(9.11) \quad \sum_{j,k=0}^n g^{jk}(x) \psi_{x_j} \varphi_{px_k} = 0 \quad \text{near } x^{(0)}, \quad 1 \leq p \leq n-1,$$

with the initial condition

$$(9.12) \quad \varphi_p(x_0, x)|_{\Gamma_2} = s_p(x), \quad 1 \leq p \leq n-1,$$

where $s_1(x), \dots, s_{n-1}(x)$ are coordinates on S_0 near $x^{(0)}$, $\frac{Dx}{D(s, s_1, \dots, s_{n-1})} \neq 0$ in B_ε . Note that φ_p does not depend on x_0 and $\psi(x_0, x) = T - x_0 + \psi_1(x)$, where $\psi_1(x)$ also does not depend on x_0 .

We shall make the change of coordinates in D_{0T}

$$(9.13) \quad \begin{aligned} s &= \varphi(x), \\ \tau &= \psi(x_0, x), \\ y_j &= \varphi_j(x), \quad 1 \leq j \leq n-1. \end{aligned}$$

Note that the Jacobian $\frac{D(x_0, x)}{D(s, \tau, y')} \neq 0$ in B_ε where $y' = (y_1, \dots, y_{n-1})$. Rewrite Lu_0 in (s, τ, y') coordinates (c.f. [E1], [E4]). We get

$$(9.14) \quad \begin{aligned} \hat{L}\hat{u}_0 &= \frac{1}{\sqrt{|\hat{g}|}} \frac{\partial}{\partial s} \sqrt{|\hat{g}|} \hat{g}^{s\tau}(s, \tau, y') \frac{\partial \hat{u}_0}{\partial \tau} + \frac{1}{\sqrt{|\hat{g}|}} \frac{\partial}{\partial \tau} \sqrt{|\hat{g}|} \hat{g}^{s\tau}(s, \tau, y') \frac{\partial \hat{u}_0}{\partial s} \\ &+ \sum_{j=1}^{n-1} \frac{1}{\sqrt{|\hat{g}|}} \frac{\partial}{\partial s} \sqrt{|\hat{g}|} \hat{g}^{sj}(s, \tau, y') \frac{\partial \hat{u}_0}{\partial y_j} + \sum_{j=1}^n \frac{1}{\sqrt{|\hat{g}|}} \frac{\partial}{\partial y_j} \sqrt{|\hat{g}|} \hat{g}^{sj}(s, \tau, y') \frac{\partial \hat{u}_0}{\partial s} \\ &+ \sum_{j,k=1}^{n-1} \frac{1}{\sqrt{|\hat{g}|}} \frac{\partial}{\partial y_j} \sqrt{|\hat{g}|} \hat{g}^{jk}(s, \tau, y') \frac{\partial \hat{u}_0}{\partial y_k} \stackrel{\text{def}}{=} \hat{L}_1 \hat{u}_0 + \hat{L}_2 \hat{u}_0, \end{aligned}$$

where L_2 is the last sum in (9.14) and L_1 are the remaining sum. Note that $\hat{g}^{ss} = \hat{g}^{\tau\tau} = \hat{g}^{\tau j} = 0$, $1 \leq j \leq n-1$, because of (9.8), (9.9), (9.11). In (9.14) $\hat{u}_0(s, \tau, y') = u_0(x_0, x)$ where (x_0, x) and (s, τ, y') are related by (9.13). Since T and ε are small we can introduce (s, τ, y') coordinates in D_{0T} . Denote by \hat{D}_{0T} the image of D_{0T} in (s, τ, y') coordinates.

Let $\partial\hat{\Sigma}_0$ be the image of $\partial\Sigma(x^{(0)})$ in (s, τ, y') coordinates. Since $u_0 = 0$ near $\partial\Sigma(x^{(0)})$ we have that $\hat{u}_0 = \hat{\alpha}_0 \hat{u} = 0$ near $\partial\hat{\Sigma}_0$. Note also that $u_0 = u_{0x_0} = 0$ for $x_0 = 0$ and we extend u_0 by zero for $x_0 < 0$. Since $\varphi_x(x^{(0)})$ is the outward normal to Ω_{int} we have $s = \varphi(x) \geq 0$ on $\overline{\Omega}_{ext}$ near S_0 . Since $\tau = \psi(T, x) = 0$ on Γ_2 and $\psi_{x_0}|_{\Gamma_2} = -1$ we have that $\tau \leq 0$ in \hat{D}_{0T} .

Denote by (\hat{u}, \hat{v}) the L^2 inner product in \hat{D}_{0T} . Consider

$$(9.15) \quad (\hat{L}\hat{u}_0 - \hat{f}_0, \hat{g}^{s\tau}\hat{u}_{0\tau} + \sum_{j=1}^{n-1} \hat{g}^{s0}\hat{u}_{0y_j} - \hat{g}^{s\tau}\hat{u}_{0s}) = 0$$

Let

$$(9.16) \quad \begin{aligned} I_1 &= (\hat{L}_1\hat{u}_0, \hat{g}^{s\tau}\hat{u}_{0\tau} + \sum_{j=1}^{n-1} \hat{g}^{s0}\hat{u}_{0y_j}) \\ &= \int_{\hat{D}_{0T}} \frac{\partial}{\partial s} (\hat{g}^{s\tau}\hat{u}_{0\tau} + \sum_{j=1}^{n-1} \hat{g}^{s0}\hat{u}_{0y_j})^2 ds d\tau dy' + Q_1(\hat{u}_0, \hat{u}_0), \end{aligned}$$

where Q_1 has an estimate

$$(9.17) \quad |Q_1(\hat{u}_0, \hat{u}_0)| \leq C \int_{\hat{D}_{0T}} (\hat{u}_{0s}^2 + \hat{u}_{0\tau}^2 + \sum_{j=1}^{n-1} \hat{u}_{0y_j}^2) ds d\tau dy'.$$

Therefore

$$(9.18) \quad I_1 = - \int_{\Gamma_2} (\hat{g}^{s\tau}\hat{u}_{0\tau} + \sum_{j=1}^{n-1} \hat{g}^{s0}\hat{u}_{0y_j})^2 d\tau dy' + Q_1(\hat{u}_0, \hat{u}_0).$$

Denote

$$(9.19) \quad I_2 = \left(\frac{1}{\sqrt{|\hat{g}|}} \frac{\partial}{\partial s} \sqrt{|\hat{g}|} \hat{g}^{s\tau}(s, \tau, y') \frac{\partial \hat{u}_0}{\partial \tau} + \frac{1}{\sqrt{|\hat{g}|}} \frac{\partial}{\partial \tau} \sqrt{|\hat{g}|} \hat{g}^{s\tau}(s, \tau, y') \frac{\partial \hat{u}_0}{\partial s}, -\hat{g}^{s\tau}\hat{u}_{0s} \right).$$

We have

$$(9.20) \quad I_2 = - \int_{D_{0T}} \frac{\partial}{\partial \tau} ((\hat{g}^{s\tau})^2 \hat{u}_{0s}^2) ds d\tau dy' + Q_2(\hat{u}_0, \hat{u}_0).$$

Therefore

$$(9.21) \quad I_2 = - \int_{\Gamma_1} (\hat{g}^{s\tau})^2 \hat{u}_{0s}^2 ds dy' + Q_2(\hat{u}_0, \hat{u}_0).$$

Denote

$$(9.22) \quad I_3 = (\hat{L}_2 \hat{u}_0, \hat{g}^{s\tau}(\hat{u}_{0\tau} - \hat{u}_{0s})).$$

Integrating by parts in y_j (note that $\hat{u}_0 = 0$ near $\partial\hat{\Sigma}_0$ and $u_0 = 0$ for $x_0 < 0$) we get

$$(9.23) \quad \begin{aligned} I_3 &= - \int_{\hat{D}_T} \sum_{j,k=1}^{n-1} \hat{g}^{s\tau} \hat{g}^{jk} (\hat{u}_{0sy_j} - \hat{u}_{0\tau y_j}) \hat{u}_{0y_k} ds d\tau dy' + Q_3 \\ &= \frac{1}{2} \int_{\Gamma_1} \left(\sum_{j,k=1}^{n-1} g^{jk} \hat{u}_{0y_j} \hat{u}_{0y_k} \right) \hat{g}^{s\tau} ds dy' + \frac{1}{2} \int_{\Gamma_2} \left(\sum \hat{g}^{jk} \hat{u}_{0y_j} \hat{u}_{0y_k} \right) \hat{g}^{s\tau} d\tau dy' + Q_4. \end{aligned}$$

Note that

$$(9.24) \quad I_4 = (\hat{L}_2 \hat{u}_0, \sum_{j=1}^{n-1} \hat{g}^{sj} \hat{u}_{0y_j}) = Q_5(\hat{u}, \hat{u})$$

since it can be represented as a divergence of quadratic form in \hat{u}_{0y_j} (c.f. (8.15)). Also

$$|(\hat{f}_0, \hat{g}^{s\tau}(\hat{u}_{0\tau} - \hat{u}_{0s}) + \sum_{j=1}^{n-1} \hat{g}^{js} \hat{u}_{0y_j})| \leq C \int_{\hat{D}_{0T}} |\hat{f}_0|^2 ds d\tau dy' + Q_6(\hat{u}_0, \hat{u}_0).$$

Therefore we have

$$(9.25) \quad \begin{aligned} &\frac{1}{2} \int_{\Gamma_2} \left(- \sum_{j,k=1}^{n-1} \hat{g}^{jk} \hat{u}_{0y_j} \hat{u}_{0y_k} \hat{g}^{s\tau} \right) d\tau dy' + \frac{1}{2} \int_{\Gamma_1} \left(- \sum_{j,k=1}^{n-1} \hat{g}^{jk} \hat{u}_{0y_j} \hat{u}_{0y_k} \hat{g}^{s\tau} \right) ds dy' \\ &\quad + \int_{\Gamma_2} (\hat{g}^{s\tau} \hat{u}_{0\tau} + \sum_{j=1}^{n-1} \hat{g}^{sj} \hat{u}_{0y_j})^2 d\tau dy' + \int_{\Gamma_1} (\hat{g}^{s\tau})^2 \hat{u}_{0s}^2 ds dy' \\ &\leq Q_7(\hat{u}_0, \hat{u}_0) + C \int_{\hat{D}_{0T}} |\hat{f}_0|^2 ds d\tau dy' \end{aligned}$$

Note that $\hat{g}^{s\tau} > 0$ and

$$(9.26) \quad - \sum_{j,k=1}^{n-1} \hat{g}^{jk} \hat{u}_{0y_j} \hat{u}_{0y_k} \geq C \sum_{j=1}^{n-1} \hat{u}_{0y_j}^2 \quad \text{in } \hat{D}_{0T}.$$

Therefore

$$(9.27) \quad \int_{\Gamma_1} [(\hat{g}^{s\tau})^2 \hat{u}_{0s}^2 - \frac{1}{2} \sum_{j,k=1}^{n-1} \hat{g}^{jk} \hat{u}_{0y_j} \hat{u}_{0y_k} \hat{g}^{s\tau}] ds dy'$$

is equivalent to $\int_{\Gamma_1} (\hat{u}_{0s}^2 + \sum_{j=1}^{n-1} \hat{u}_{0y_j}^2) ds dy'$, i.e. it is equivalent to the norm $\|\hat{u}_0\|_{1,\Gamma_1}^2$ in $H^1(\Gamma_1)$. Analogously

$$(9.28) \quad \int_{\Gamma_2} [(\hat{g}^{s\tau} \hat{u}_{0\tau} + \sum_{j=1}^{n-1} \hat{g}^{sj} \hat{u}_{0y_j})^2 - \frac{1}{2} \sum_{j,k=1}^{n-1} \hat{g}^{jk} \hat{u}_{0y_j} \hat{u}_{0y_k} \hat{g}^{s\tau}] d\tau dy'$$

is equivalent to the norm

$$\|u_0\|_{1,\Gamma_2}^2 = \int_{\Gamma_2} (\hat{u}_{0\tau}^2 + \sum_{j=1}^{n-1} \hat{u}_{0y_j}^2) d\tau dy'$$

in $H^1(\Gamma_2)$. Therefore (9.25) is equivalent to

$$(9.29) \quad \|\hat{u}_0\|_{1,\Gamma_1}^2 + \|\hat{u}_0\|_{1,\Gamma_2}^2 \leq Q_8(\hat{u}_0, \hat{u}_0) + C \int_{\hat{D}_{0T}} |\hat{f}_0|^2 ds d\tau dy'.$$

Denote by $D_{0T,t}$ the intersection of D_{0T} with the half-space $x_0 \geq t$. Integrating by parts in the integral

$$(9.30) \quad 0 = \int_{D_{0T,t}} (Lu_0 - f_0)(g^{00}u_{0x_0} + \sum_{j=1}^n g^{0j}(x)u_{0x_j}) dx dx_0$$

we obtain (c.f. [E1], [E3])

$$(9.31) \quad \begin{aligned} & \int_{D_{0T} \cap \{x_0=t\}} [(\sum_{j=0}^n g^{j0}u_{0x_j}(t, x))^2 \\ & + (\sum_{j=1}^n g^{j0}u_{0x_j}(t, x))^2 - \sum_{j,k=1}^n g^{jk}u_{0x_j}(t, x)u_{0x_k}(t, x)] dx \\ & \leq C(\|u_0\|_{1,\Gamma_{1t}}^2 + \|u_0\|_{1,\Gamma_{2t}}^2) + C \int_{D_{0T,t}} (\sum_{j=0}^n u_{0x_j}^2) dx dx_0 + C \int_{D_{0T,t}} |f_0|^2 dx dx_0, \end{aligned}$$

where Γ_{jt} is the intersection of Γ_j with $x_0 \geq t$, $j = 1, 2$. Note that the integral in the left hand side of (9.31) is equivalent to

$$(9.32) \quad \int_{D_{0T} \cap \{x_0=t\}} \left(\sum_{j=0}^n u_{0x_j}^2(t, x) \right) dx.$$

Rewriting (9.29) in (x_0, x) coordinates and combining with (9.31) we get

$$(9.33) \quad \max_{0 \leq t \leq T} \int_{D_{0T} \cap \{x_0=t\}} \left(\sum_{j=0}^n u_{0x_j}^2 \right) dx \leq C \int_{D_{0T}} \left(\sum_{j=0}^n u_{0x_j}^2 \right) dx_0 dx + C \int_{D_{0T}} |f_0|^2 dx_0 dx.$$

Let $\{\alpha_j(x)\}_{j=1, \dots, N}$ be as above. Denote $u_j = \alpha_j u$. Applying (9.33) with $u_j = \alpha_j u$ instead of $u_0 = \alpha_0 u$ and using that $\sum_{j=1}^N \alpha_j = 1$ in D_T we get

$$(9.34) \quad \begin{aligned} & \max_{0 \leq t \leq T} \int_{D_T \cap \{x_0=t\}} \left(\sum_{j=0}^n u_{x_j}^2(t, x) \right) dx \\ & \leq C \int_{D_T} \left(\sum_{j=0}^n u_{x_j}^2(x_0, x) \right) dx_0 dx + C \sum_{j=1}^N \int_{D_{jT}} |f_j|^2 dx_0 dx. \end{aligned}$$

where $f_j = (L\alpha_j - \alpha_j L)u$,

$$(9.35) \quad |f_j| \leq C \sum_{j=0}^n |u_{x_j}|.$$

Therefore

$$(9.36) \quad \begin{aligned} & \max_{0 \leq t \leq T} \int_{D_T \cap \{x_0=t\}} \left(\sum_{j=0}^n u_{x_j}^2(t, x) \right) dx \\ & \leq C \int_{D_T} \left(\sum_{j=0}^n u_{x_j}^2(x_0, x) \right) dx_0 dx \leq CT \max_{0 \leq t \leq T} \int_{D_T \cap \{x_0=t\}} \left(\sum_{j=0}^n u_{x_j}^2(t, x) \right) dx \end{aligned}$$

Since T is small we conclude that $u = 0$ in D_T . Take any $T_1 < T$. Then there exists $\delta_1 > 0$ such that $S_{0\delta_1} \times [0, T_1] \subset D_T$ where $S_{0\delta_1}$ is a δ_1 -neighborhood of S_0 . Therefore $u(x_0, x) = 0$ in $S_{0\delta_1} \times [0, T_1]$ and we can extend $u(x_0, x)$ by zero in $\Omega_{int} \times [0, T_1]$. Then $Lu = 0$ in $\mathbf{R}^n \times (0, T_1)$ and $u(0, x) = u_{x_0}(0, x) = 0$ in \mathbf{R}^n . By the uniqueness of the hyperbolic Cauchy problem (c.f. (9.5)) we have

$u = 0$ in $\mathbf{R}^n \times (0, T_1)$. Repeating the same arguments on $(T_1, 2T_1)$, etc., we get that $u = 0$ in $\Omega_{ext} \times (0, T)$ for any $T > 0$. Therefore $v = \hat{u}$ in $\Omega_{ext} \times (0, T)$ where $v(x_0, x)$ satisfies (9.1), (9.2) and \hat{u} satisfies (9.4) in $\mathbf{R}^n \times (0, T)$. Then (9.5) implies that

$$(9.37) \quad \begin{aligned} & \max_{0 \leq x_0 \leq T} (\|v(x_0, \cdot)\|_{1, \Omega_{ext}}^2 + \|v_{x_0}(x_0, \cdot)\|_{0, \Omega_{ext}}^2) \\ & \leq C(\|\varphi_0\|_{1, \Omega_{ext}}^2 + \|\varphi_1\|_{0, \Omega_{ext}}^2 + \int_0^T \|f(x_0, \cdot)\|_{0, \Omega_{ext}}^2 dx_0). \end{aligned}$$

Therefore we proved an analogue of Theorem 8.3:

Theorem 9.1. *Let S_0 be a characteristic surface inside the ergosphere S_Δ and let $v(x_0, x)$ satisfies (9.1), (9.2). Suppose (5.3) holds. Then $v(x_0, x)$ satisfies (9.37).*

Note that (9.37) implies that $D_+(\Omega_{int} \times \mathbf{R}) \subset \overline{\Omega}_{int} \times \mathbf{R}$, i.e. that $\Omega_{int} \times \mathbf{R}$ is a black hole.

As in the case of Theorem 8.1 we can take into account boundary condition on $\partial\Omega_0$ and prove that estimate of the form

$$(9.38) \quad \begin{aligned} & \max_{0 \leq x_0 \leq T} (\|v(x_0, \cdot)\|_{1, \Omega_{ext} \cap \Omega}^2 + \|v_{x_0}(x_0, \cdot)\|_{0, \Omega_{ext} \cap \Omega}^2) \\ & \leq C \int_0^T \|f(x_0, \cdot)\|_{0, \Omega_{ext} \cap \Omega}^2 dx_0 + [g]_{1, \partial\Omega_0 \times (0, T)}^2. \end{aligned}$$

When (5.2) holds we get that the domain D_T will be contained in $\Omega_{int} \times (0, T)$. In this case the proof similar to the proof of Theorem 9.1 gives an estimate of the form (9.38) in $(\Omega_{int} \cap \Omega) \times (0, T)$, i.e. in this case $\Omega_{int} \times \mathbf{R}$ is a white hole.

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